What to Expect when Everyone is Expecting: Self-Fulfilling Expectations and Asset-Pricing Puzzles

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Abstract

We construct equilibria of continuous-time, overlapping generations economies whereby interest rates and asset prices affect the distribution of wealth and consumption between existing and arriving cohorts of investors. Such economies are prone to self-fulfilling expectations and a multitude of equilibria; anticipations of future discount rates impact asset prices and the wealth distribution, causing savings and investment responses that end up confirming the discount rate anticipations. Extrinsic uncertainty caused by shifts in expectations about the prevailing equilibrium imply volatility that is entirely disconnected from “fundamentals.” Moreover, this volatility implies a non-trivial risk premium for stocks even in a world where aggregate consumption follows a deterministic time trend and investors have standard recursive preferences with only moderate degrees of risk aversion.

Keywords: asset pricing, self-fulfilling expectations, sunspot equilibria, equity premium puzzle, excess volatility puzzle, inequality

JEL Classification: G01, G12
1 Introduction

Anticipations of future discount rates can impact the wealth distribution between existing and arriving cohorts of investors depending on how the value of their endowments is affected by variations in discount rates. In the textbook, representative agent model such redistribution is irrelevant since all agents are members of the same family, connected through bequest and gift motives. If, however, investors cannot be consolidated into a single representative family, then anticipations of future discount rates can become a self-fulfilling prophecy if they affect the consumption behavior of existing and arriving investors in different ways. For instance, if anticipations of low future discount rates induce arriving cohorts to consume more (by borrowing against the increased present value of their human capital), then this anticipation becomes self-fulfilling: As arriving agents consume a larger fraction of aggregate consumption, existing cohorts experience a declining anticipated consumption, inducing them to save more and in the end confirming the anticipation of low discount rates.

We illustrate such feedback effects in an intentionally stylized overlapping generations model. In the model the arriving cohort of consumers is endowed with a depreciating endowment process. We interpret such a depreciation as the result of obsolescence of both non-traded human capital and also of the rents of existing firms, which are displaced by arriving rivals. We show that already this intentionally stylized model features multiple equilibria. Moreover, it is possible that extrinsic uncertainty, i.e., uncertainty caused by fluctuations in some exogenous signal that is perceived as a coordination device, can impact the relative valuation of financial and human wealth, leading to fluctuations in asset prices that are unrelated to fundamentals. We illustrate this point by showing that volatility of financial wealth arises even in cases where aggregate consumption and dividends of existing firms are deterministic.

In this motivating, stylized model the multiplicity of equilibria arises when and only when financial wealth includes a rational bubble. A novel aspect of our framework is that a bubble is actually not required for multiple equilibria to exist. We show this by considering a slightly more general model where the arrival rate of new firms — and accordingly the obsolescence rate of existing ones — depends on the relative price of human capital and financial wealth. (We show that an elementary risk-return choice by arriving entrepreneurs
will naturally imply such dependence on this relative price). In such a situation equilibrium multiplicity survives, but without the need for bubbles (or for an interest rate that is below the aggregate growth rate). The key common property of the two models is that fluctuations in aggregate financial wealth and aggregate human capital offset each other, rendering the composition of total wealth (but not the sum) indeterminate.

The indeterminacy of the composition of aggregate wealth may be a source of volatility of financial wealth, but the associated fluctuations may not carry a risk premium. The reason is that existing agents’ consumption processes must be locally predictable in a world where aggregate consumption is deterministic. Accordingly, there can be no risk premium associated with such fluctuations if investors have expected utility preferences. If however, investors have recursive preferences, this extrinsic uncertainty carries a risk premium since investors care about the long-run impact of the relative asset fluctuations on their wealth, which is not predictable and carries a risk premium.

From a practical perspective, introducing extrinsic uncertainty is particularly useful in terms of quantitatively addressing asset-pricing puzzles. A very common challenge of such models is that the volatility of asset prices has to be driven by volatility in fundamentals (consumption, dividends, etc.) which typically exhibit substantially lower volatility than asset prices. The logic of the calibration of such models is to take the small macroeconomic volatility as given (e.g., the small volatility of aggregate consumption growth), and introduce some magnification mechanism (e.g., a substantially non-linear and time-varying risk aversion) to match asset pricing models. In our model, there is no intrinsic uncertainty, and so both asset prices and macroeconomic quantities (e.g., individual consumption growth) are jointly endogenous to shifts in self-fulfilling expectations. This subtle difference introduces a crucial degree of freedom, which allows us to reverse the sequence of calibration: We calibrate the expectational shifts to match the volatility of asset prices and then examine the implications of the model for the real side of the economy (e.g., for individual consumption processes).

We calibrate the model and show that we are able to obtain realistic risk premia and volatility even when consumption and dividends have no quadratic variation, investors have unitary inter temporal elasticity of substitution and very moderate risk aversion (around 5-6). Equally importantly, the shocks that we consider imply very muted fluctuations in the
cross sectional consumption distribution. Our main conclusion is that changing the nature of shocks to be non-fundamental, which allows us to reverse the sequence of calibration, makes it much less challenging to match asset pricing puzzles, while obtaining realistic implications for the real side of the economy.

Finally, the paper also makes a technical contribution. Due to the continuous-time setup, the construction of equilibria subject to extrinsic uncertainty becomes particularly tractable. With judicious choices of the drift and volatility for the variables governing extrinsic uncertainty, it becomes possible to provide simple closed-form solutions for prices, interest rates, equity premia, etc., which greatly facilitates the calibration and simulation of the model. More generally, one can typically provide a model solution in terms of a single, second-order, differential equation, which can be readily solved numerically.

1.1 Relation to the literature
To be written.

2 Baseline Model
We start by presenting an intentionally stylized, overlapping generations model in continuous time. The main goal of this model is to lay out the economic intuitions that we build on in later sections.

2.1 Consumers
Time is continuous. Each agent faces a constant hazard rate of death $\lambda > 0$ throughout her life, so that a fraction $\lambda$ of the population perishes at each instant. A new cohort of mass $\lambda$ is born per unit of time, so that the total population remains at $\lambda \int_{-\infty}^{t} e^{-\lambda(t-s)} ds = 1$. For the presentation of the baseline model, we assume that consumers maximize

$$\int_{s}^{\infty} e^{-\rho(t-s)} \log (c_{t,s}) ,$$

(1)
where \( s \) is the time of their birth and \( t \) is calendar time. The assumption of logarithmic preferences implies a unitary intertemporal elasticity of substitution (IES), which facilitates simple, closed-form solutions; in later sections, we introduce general, recursive preferences.

Consumers have no bequest (or gift) motives for simplicity.

### 2.2 Firms

Output is produced by a continuum of firms indexed by the time of their birth \( s \). A (representative) firm born at time \( s \) produces output

\[
y_{t,s} = e^{-\delta(t-s)} A_s, \quad \delta \geq 0
\]

where the exponential term \( e^{\delta(s-t)} \) captures technological obsolescence as calendar time \( t \) increases, and \( A_s \) captures a deterministically growing technological “frontier”, which grows exogenously at the rate \( g \geq 0 \):

\[
\frac{\dot{A}_t}{A_t} = g dt.
\]

At each time \( t \) a fraction \( \alpha \) of \( y_{t,s} \) accrues to all the agents born at time \( s \) as (non-tradable) labor income. The remaining fraction \( 1 - \alpha \) accrues to the time-\( t \) shareholders as dividends.

A specification such as (1) arises naturally in overlapping-generations models where firms are viewed as embodying new ideas for producing consumption goods, and are rivalrous to previous firms. (See, e.g., Garleanu, Kogan, Panageas (2012), Caballero, Farhi, Gourinchas (2015).) Figure 1 illustrates the production function (2).

The assumption that a fraction of firms’ income accrues as wage income to cohort-\( s \) agents would arise naturally if one assumed that workers of cohort \( s \) have skills that are specific to firms created at time \( s \). This assumption is purely for convenience and is relaxed later.

We assume that new firms arrive at the rate \( g + \delta \), which implies that total output normalized by \( A_t \) is equal to

\[
\frac{Y_t}{A_t} = \int_{-\infty}^{t} \frac{y_{t,s}}{A_t} ds = (g + \delta) \int_{-\infty}^{t} e^{-(g+\delta)(t-s)} = 1.
\]
The cohort of consumers born at time $t$ are the (initial) owners of the firms that enter the market at time $t$. They sell these firms to the market and use the proceeds (along with their wages income) to finance their life-time consumption plans.

## 2.3 Markets

Markets are dynamically complete. Investors can trade in instantaneously maturing riskless bonds in zero net supply, which pay an interest rate $r_t$. Consumers can also trade claims on all existing firms (normalized to unit supply). Finally, investors can access a market for annuities through competitive insurance companies as in Blanchard (1985), allowing them to receive an income stream of $\lambda W_{t,s}$ per unit of time, where $W_{t,s}$ is the consumer’s financial wealth. In exchange, the insurance company collects the agent’s financial wealth when she dies. Entering such a contract is optimal for all agents, given the absence of bequest motives.
A consumer’s dynamic budget constraint is given by

\[ dW_{t,s} = (r_t + \lambda) W_{t,s} dt + w_{t,s} dt + \int_{-\infty}^{t} \{ \theta_{t,s} (dP_{t,s} + \alpha y_{t,s} dt - r_t dt) \} ds, \]  

where \( W_{t,s} \) is a consumer’s wealth, \( P_{t,s} \) is the value of the representative firm of vintage \( s \), \( \theta_{t,s} \) is the number of shares of each company, and \( w_{t,s} = (1 - \alpha) \frac{y_{t,s}}{\lambda e^{-\lambda(t-s)}} \) is per-capita labor income.

2.4 Equilibrium

The equilibrium definition is standard. We look for consumption processes \( c_{t,s} \), asset allocations \( \theta_{t,s} \), asset prices \( P_{t,s} \), and an interest rate \( r_t \) such that a) consumers maximize (1) subject to (3), b) the goods market clears, i.e., \( \lambda \int_{-\infty}^{t} e^{-\lambda(t-s)} c_{t,s} ds = Y_t \), c) assets markets clear, i.e., \( \int_{-\infty}^{t} \lambda e^{-\lambda(t-s)} \theta_{t,s} ds = 1 \) and \( \int_{-\infty}^{t} \lambda e^{-\lambda(t-s)} (W_{t,s} - \theta_{t,s} P_{t,s}) ds = 0. \)

3 Baseline Model: Solution and Analysis

The model presented thus far is a “bare bones” overlapping-generations, endowment economy with continuously arriving firms and consumers. Importantly, there are no fundamental shocks in the model. In this section we derive a deterministic equilibrium to the model, which helps us illustrate the presence of multiple equilibria. We also use this simple model to introduce stochastic equilibria capturing shifts in the manner in which investors coordinate expectations.

We start with some definitions. Throughout, we use \( \hat{C}_t, \hat{Y}_t \), etc., as short hand for the de-trended values \( \hat{C}_t, \hat{Y}_t \), etc. We also introduce the constants

\[ \beta \equiv \lambda + \rho \]
\[ \eta \equiv \delta + g, \]

to simplify notation. Finally, we have the following definition.

**Definition 1**: Let \( q_{t,s} \) denote the ratio of the present value of all future \( y_{u,s} \) divided by the
current $y_{t,s}$:

$$q_{t,s} \equiv \frac{E_t \int_t^\infty e^{-\int_t^u r_{s,v} d\nu_{s,v}} d\nu_{u,s}}{y_{t,s}}.$$  \hspace{1cm} (4)

We note that $q_{t,s}$ is independent of $s$, since $\frac{y_{u,s}}{y_{t,s}}$ is not a function of $s$. Accordingly, we shall write $q_t$ rather than $q_{t,s}$.

To construct an equilibrium, we start from the deterministic Euler equation:

$$\frac{\dot{c}_{t,s}}{c_{t,s}} = -(\rho - r_t)$$  \hspace{1cm} (5)

for all $s \leq t$. Using (5) together with the definition of aggregate consumption $C_t = \lambda \int_{-\infty}^t e^{-\lambda(t-s)} c_{t,s} ds$ implies that

$$\dot{\hat{C}}_t = -(\lambda + g) \hat{C}_t + \lambda \int_{-\infty}^t e^{-\lambda(t-s)} \dot{c}_{t,s} + \lambda \gamma_{t,t} = -(\beta + g - r_t) \hat{C}_t + \lambda \frac{c_{t,t}}{A_t}.$$  \hspace{1cm} (6)

Market clearing implies $\hat{C}_t = \hat{Y}_t = 1$ and accordingly $\hat{C}_t = 0$. Therefore (6) leads to

$$r_t = g + \beta - \lambda \frac{c_{t,t}}{A_t}.$$  \hspace{1cm} (7)

Assuming that the value of a newly arriving firm equals the present value of its dividends, we have the following Lemma.

**Lemma 1** Suppose that a newly-arriving firm's price is $P_{t,t} = \alpha y_{t,t} q_t$. Then a newly-born agent's consumption is given by

$$\frac{c_{t,t}}{A_t} = \beta \frac{1}{\lambda} \frac{y_{t,t}}{A_t} = \beta \eta \frac{1}{\lambda} q_t.$$  \hspace{1cm} (8)

Time-differentiating (4) shows that $q_t$ satisfies the asset pricing equation

$$\frac{\dot{q}_t}{q_t} - \delta + \frac{1}{q_t} = r_t.$$  \hspace{1cm} (9)

Finally, substituting (8) into (7) and the resulting expression into (9), and re-arranging
shows that $q_t$ satisfies the following ordinary differential equation

$$\dot{q}_t = (\beta + \eta) q_t - \eta \beta q_t^2 - 1.$$ (10)

Figure 2 helps illustrate the differential equation (10). Since the right-hand side of (10) is quadratic, there are two values of $q_t$ such that $\dot{q}_t = 0$ (steady states). They are given by $q^* = \frac{1}{\eta}$ and $q^{**} = \frac{1}{\beta}$. In these steady states, equations (7) and (8) imply that the interest rate would be $r^* = g$ and $r^{**} = g + \beta - \eta$ respectively.

To discuss whether these two roots $q^*$ and $q^{**}$ correspond to long term equilibria, we distinguish two cases, namely $\beta > \eta$ and $\beta < \eta$.

We start with the case $\beta > \eta$. In this case, there exists a unique long-run equilibrium and it is associated with $q_t = q^{**}$. To see why only $q_t = q^{**}$ is an equilibrium, we compute the value of the stock market. The assumption of logarithmic preferences implies a constant consumption-to-total-wealth ratio of $\beta$. Accordingly, total wealth in the economy is $C_t = \frac{C_t}{\beta}$. Subtracting the total value of human capital implies that total financial wealth is given by $C_t - (1 - \alpha) q_t Y_t$. In equilibrium it must be the case that $C_t = Y_t$ and that the value of the stock market $P_t$ equals the total financial wealth of households (since bonds are in zero net supply). Accordingly, it must be the case that

$$\frac{P_t}{Y_t} = 1 \frac{1}{\beta} - (1 - \alpha) q_t.$$ (11)

If $q_t = q^{**} = \frac{1}{\beta}$, then the above equation implies $P_t = \alpha q_t Y_t$, which is the particular solution of the asset-pricing equation (9). Hence, $q_t = q^{**}$ and $P_t = \alpha q^{**} Y_t$ corresponds to a long run equilibrium. If, however, $q_t = q^*$, then (11) implies

$$P_t = \left( \frac{1}{\beta} - (1 - \alpha) \frac{1}{\eta} \right) Y_t < \alpha \frac{1}{\eta} Y_t = \alpha q^* Y_t,$$ (12)

where the inequality follows from the assumption $\beta > \eta$. In words, the value of the stock market would have to be lower than the present value of dividends. This could only happen if the aggregate stock market contained a negative bubble growing at the rate $r^* = g > 0$, which would eventually be inconsistent with free asset disposal.\footnote{Recall that the fundamental value of an existing firm declines at the rate $\delta$, so that eventually the sum}
Hence the only long-run equilibrium is given by $q_t = q^{**} = \frac{1}{\beta}$ when $\beta > \eta$, and $P_t = \alpha q^{**}Y_t$ is the only market clearing price for all $t$.

The situation is different if $\beta < \eta$. In this case there are two long run equilibria. The first equilibrium is given by $q_t = q^{**}$ and accordingly $r_t = r^{**} = g + \beta - \eta < g$. Following the same reasoning as above, in this equilibrium $P_t = \alpha_qY_t$ and hence markets clear.

The case $q_t = q^*$ is also a long run equilibrium. Because $\beta < \eta$, inequality (12) is reversed so that $P_t > \alpha q^*Y_t$. In this case, the price $P_t$ contains a positive bubble given by $b_t = P_t - \alpha q^*Y_t = \left(\frac{1}{\beta} - \frac{1}{\eta}\right)Y_t$. Since $r^* = g$ and $Y_t$ grows that the rate $g$, it follows that $\dot{b}_t = rb_t$, so that investors are actually willing to hold the bubble that is attached to the price of existing assets. The next proposition generalizes this observation.

**Proposition 1** Suppose that $\beta < \eta$. For $q_{t_0} \in (q^*, q^{**}]$, define $P_{t_0} = \left(\frac{1}{\beta} - q_{t_0}\right)Y_{t_0}$. Take $q_t$ that solves (10) subject to the initial condition $q_t(t_0) = q_{t_0}$ and define $b_{t_0} \equiv P_{t_0} - \alpha q_{t_0}Y_{t_0}$. Then $\dot{b}_t = r_t b_t$ for all $t > t_0$ and $P_t = \left(\frac{1}{\beta} - q_t\right)Y_t$ is a market clearing price for all $t \geq t_0$.

The right plot of figure 2 illustrates Proposition 1. From any choice of initial $q_{t_0} \in (q^*, q^{**}]$ there originates a path $q_t$ that tends to $\lim_{t \to \infty} q_t = q^{**}$. There is an indeterminacy as to the initial choice of $q_{t_0}$. Any initial choice of $q_t \in (q^*, q^{**}]$ is associated with a self-fulfilling, perfectly anticipated path of interest rates and values of $q_t$, leading eventually to the steady of the fundamental value and the bubble attached to that firm would become negative.
state value $q^{**}$. This indeterminacy is driven by self-fulfilling beliefs about future interest rates. The force that makes these anticipations self-fulfilling is the redistributive effect that interest rates (and the associated prices) have on different cohorts. For instance, an anticipation of low interest rates in the future increases $q_t$ and the value of arriving generations’ initial endowment. This increase in their wealth raises the fraction of consumption accruing to arriving cohorts and accordingly lowers the anticipated consumption growth of existing cohorts. The lower anticipated growth consumption growth rate of existing cohorts induces them to save more, lowering interest rates (across the term structure) and hence rendering the anticipation of lower interest rates rational.

The different equilibrium paths associated with different values $q_0$ feature different initial bubbles $b_{t_0}$, which arise on the prices of existing stocks. We note, parenthetically, that the bubble could arise on assets that pay no dividends. It does not have to be attached to existing stocks.

In all equilibria that emanate from $q_{t_0} \in (q^*, q^{**}]$ the bubble grows more slowly than aggregate consumption $C_t$ (and aggregate market capitalization $P_t$). The reason is that $r_t < g$ on any of these equilibria. An exception is the equilibrium that emanates from $q_{t_0} = q^*$, in which $r = g$ and the bubble remains a constant fraction of the aggregate economy.

For the purposes of the analysis that follows, it is important to highlight one aspect of the many equilibria that arise: The sole purpose of the bubble is to introduce a component of financial wealth that decouples aggregate wealth (which is always equal to $C_t^{\frac{\gamma}{\beta}}$, due to the assumption of a unitary elasticity of substitution) from the present value of dividends and labor income owned by existing households ($q_t C_t$). This may create the impression that bubbles are essential to support the multiple equilibria in $q_t \in (q^*, q^{**}]$, but that is not the case. The important features is that, even though the magnitude of total wealth is determinate, its composition is indeterminate. Because in the simple model studied so far there is only one depreciating tree, the second asset that “offsets” $q_t$ so as to keep total wealth fixed must necessarily be a bubble, which can only arise if $r_t \leq g$. In Section 4 we show that the indeterminacy of the components of aggregate wealth also arises in a slightly more complex economy without bubbles, but with multiple assets priced at fundamental values.
3.1 Prices of arriving assets

So far we focused on equilibria in which arriving assets are priced at their fundamental value. We next relax this assumption. As long as \( \beta < \eta \), we can construct equilibria in which arriving assets are priced above the fundamental value. Equation (8) becomes

\[
\hat{c}_{t,t} = \beta \frac{1}{\lambda} \left( \eta q_t + \hat{b}_t \right),
\]

where \( \hat{b}_t = \hat{P}_{t,t} - \alpha \eta q_t \) is the initial value of the bubble for a newly arriving asset. Accordingly, equation (10) becomes

\[
\dot{q}_t = \left( \beta + \eta - \eta \beta \hat{b}_t \right) q_t - \eta \beta q_t^2 - 1.
\]

(13)

Focusing on equilibria where \( \hat{b}_t = \tilde{b} > 0 \) for all \( t \), we have the following straightforward result.

**Lemma 2** For any \( \tilde{b} < \frac{\beta + \eta - \sqrt{4 \eta \beta}}{\eta \beta} \) there exist two steady states \( q^{(1)} \) and \( q^{(2)} \) with \( q^{(1)} > \frac{1}{\eta} \) and \( q^{(2)} < \frac{1}{\beta} \). In either of these equilibria the total detrended value of the bubble in the economy is positive in the long run.

Figure 3 illustrates Lemma 2. A practical consequence of Lemma 2 is that the aggregate value of the bubble does not disappear asymptotically in either steady state. Initial values of \( q_0 \) in the range \((q^{(1)}, \frac{1}{\beta})\) all lead to the same steady state value \( q^{(2)} \) and the same value of the aggregate stock market \( \hat{P}_t = \left( \frac{1}{\beta} - (1 - \alpha) q^{(2)} \right) Y_t \). The main difference when bubbles are attached to arriving assets is that the aggregate value of the bubble does not become a vanishingly small fraction of the aggregate economy.

3.2 Extrinsic uncertainty and stochastic equilibria

The practical implication of price indeterminacy is the possibility to construct stochastic, rational expectations equilibria with the property that the stock market price is subject to self-fulfilling shifts in expectations about the future evolution of asset prices. To illustrate such equilibria, we let \( x_t \) be some Markovian diffusion with characteristics \( \mu(x_t) \) and \( \sigma(x_t) \). The process \( x_t \) is simply some signal that helps investors coordinate their actions. We will
construct equilibria whereby if investors believe that $x_t$ matters for the dynamics of $r_t$, $q_t$, etc. (so that we can write $r_t = r(x_t), q_t = q(x_t)$), and make their consumption and investment decisions accordingly, the resulting market clearing interest rates and prices are indeed given by $r_t = r(x_t), q_t = q(x_t)$, etc..

To illustrate the construction, we start by postulating a given volatility process for $\sigma(x_t)$. We normalize the support of $x_t$ to be the unit interval $[0, 1]$ and let the volatility function be given by

$$\sigma(x_t) = \sigma \sqrt{x_t (1 - x_t)}.$$  \hfill (14)

Goods market clearing implies that even in a stochastic equilibrium, existing agents' consumption is still locally deterministic.\(^2\) Accordingly, equation (6) continues to hold and therefore

$$r(x_t) = g + \beta - \beta \left( \eta q(x_t) + \hat{b}(x_t) \right).$$  \hfill (15)

\(^2\)This follows because aggregate consumption grows deterministically at the rate $g$. Even though the consumption rate of arriving consumers fluctuates, the consumption growth of existing consumers cannot have a diffusion component, since newly arriving consumers account for a measure zero of consumption.
Since investors have expected utility preferences and their consumption is locally deterministic, there can be no risk premia. Accordingly, the stochastic version of equation (9) is simply

$$\frac{1}{2} \sigma^2 (x_t) q'' (x_t) + \mu (x_t) q' (x_t) + 1 = (r (x_t) + \delta) q (x_t).$$  \hfill (16)

Substituting (15) into (16) and re-arranging leads to

$$\frac{1}{2} \sigma_t^2 q''_t + \mu_t q'_t + 1 = \left( \eta + \beta - \beta \left( \eta q_t + \tilde{b}_t \right) \right) q_t,$$  \hfill (17)

where we have adopted the notation $\sigma_t$, $\mu_t$, etc. as shorthand for $\sigma (x_t)$, $\mu (x_t)$, etc.

Motivated purely by considerations of tractability we next choose $\mu_t$ so that the resulting solution of (17) is log-linear, i.e., $q_t = q_0 e^{\nu x_t}$ for some constants $q_0$ and $\nu$. We choose $q_0$ to be some number in the range $(q^{(1)}, q^{(2)})$ and $\nu$ so that $q_0 e^{\nu} = \frac{1}{\beta}$. Furthermore we let $\tilde{b}_t = \tilde{b}$ for some sufficiently small $\tilde{b}$. Then, using (17) to solve for $\mu_t$ results in

$$\mu (x_t) = \frac{1}{\nu} \left[ (\eta + \beta - \beta (\eta q_0 e^{\nu x_t} + \tilde{b})) - \frac{1}{q_0} e^{-\nu x_t} - \frac{1}{2} \sigma^2 \nu^2 x_t (1 - x_t) \right].$$  \hfill (18)

**Lemma 3** If $q(0) \in (q^{(1)}, q^{(2)})$ and $q(1) = q_0 e^{\nu}$ then $\mu (0) > 0$ and $\mu (1) < 0$. Moreover if the constant $\sigma$ is chosen so that both $\frac{2\mu (0)}{\sigma^2} > 1$ and $-\frac{2\mu (1)}{\sigma^2} > 1$, then the process $x_t$ is stationary.

Lemma 3 implies that $x_t$ is stationary since at the boundary $x_t = 0$ the variance vanishes and the drift is positive, while at the boundary $x_t = 1$ the variance vanishes and the drift is negative.

We then have the following proposition.

**Proposition 2** Consider a diffusion $x_t$ with volatility given by (14) and drift given by (18). Then there exists a stochastic equilibrium whereby $q = q_0 e^{\nu x_t}$, the equilibrium interest rate is given by $g + \beta - \beta (\eta q (x_t) + \tilde{b})$, and the equilibrium price of the stock market is given by $P (x_t) = \frac{1}{\beta} - (1 - \alpha) q (x_t)$.

The stochastic equilibria of Proposition 2 rely on the fact that the composition of aggregate wealth between financial wealth $P_t$ and the value of human capital $(1 - \alpha) q (x_t)$
depends on anticipations of future interest rates, which become self-fulfilling by impacting the endowment of arriving generations and hence the consumption growth rate of existing generations.

A limitation of the stochastic equilibria constructed in Proposition 2 is that since consumption of existing investors has no quadratic variation (i.e., zero diffusion term), equity premia are zero. This conclusion ceases to hold if investors have recursive preferences as we show in Section 5, where we derive the equity premium resulting from the present model.

4 Full Model: Multiple Assets and No Bubbles

The previous section considers a simple model, meant to illustrate that the composition of aggregate wealth can be indeterminate in overlapping generations economies. To keep the analysis as tractable as possible, the focus is on an example where aggregate wealth consists of human capital \((1 - \alpha) q_t Y_t\) and financial wealth \(P_t = b_t + \alpha q_t Y_t\), with \(b_t\) a bubble.

The cost of the simplicity of the previous setup is that the analysis is necessarily limited to cases in which bubbles can exist, i.e., to cases in which \(r_t < g\). Furthermore, both the plausibility and the empirical relevance of rational bubbles is subject to debate.

In this section we consider an economy where a) labor income and dividend income are subject to different degrees of depreciation, and b) the creation of new firms is endogenous to the relative price of the two income streams.

Specifically, we modify the previous setup as follows. At birth, agents are of two types, workers or entrepreneurs. Workers receive labor income over their life. Entrepreneurs introduce a new firm into the market, which embodies the new technological frontier \(A_t\) and also steals business from existing trees.

The labor income at time \(t\) of a worker born at time \(s\) is given by

\[
  w_{t,s} = \frac{(1 - \alpha) (\delta^l + g) A_s e^{-\delta^l(t-s)}}{l_{t,s}},
\]

where \(l_{t,s} = \lambda (1 - \varepsilon_s) e^{-\lambda (t-s)}\). In the above expression \(\delta^l\) captures the obsolences rate of the cohort’s labor skills over time, \(1 - \varepsilon_s\) is the fraction of cohort-\(s\) agents who chose to become workers. Aggregating over all workers in the economy implies that total labor
The total dividend income produced by firms of cohort \( s \) at time \( t \) is

\[
D_{t,s} = \alpha (\delta^d_s + g) A_s e^{-\int_t^s \delta^d_t \delta_t^d dt}.
\]  

(20)

Specification (20) captures three aspects. First, total dividend income is a constant fraction of \( A_t \), since \( \int_{-\infty}^t D_{t,s} ds = \alpha A_t \). Second, new firms embody new technologies \( (A_t) \), and therefore are a source of growth; however, they also displace the dividends of existing firms as in the baseline model. Third, and in contrast to the baseline model, we endogenize the total dividends \( D_{t,t} \) accruing to arriving firms as follows.

The entrepreneurs, who constitute a fraction \( \varepsilon < 1 \) of arriving agents at time \( s \), are faced with two choices: a) introduce a company that produces dividends \( \alpha \psi A_s e^{-\int_t^s \delta_t^d \delta_t^d dt} \) for times \( t \geq s \), (the “safe” choice) or b) introduce a company that is successful with probability \( \pi \in (0, 1) \) and produces a dividend \( \alpha \xi_i A_s e^{-\int_t^s \delta_t^d \delta_t^d dt} \) for \( t \geq s \), or is worthless with probability \( 1 - \pi \) (the “risky” choice). If the firm ends up being worthless, the entrepreneur must then become a worker for the remainder of her life, so as to finance a positive consumption stream. The choice between the safe and the risky option happens once, at birth. Without loss of generality, we assume that \( \xi : [0, \varepsilon] \to \mathbb{R}^+ \) is a decreasing function. To make matters interesting \( \pi \xi (\varepsilon) > \psi \), so that there is a risk-return trade-off.

If a measure \( \zeta \) of entrepreneurs choose the risky option and a measure \( \varepsilon - \zeta \) choose the secure option, then

\[
D_{t,t} = \alpha A_t (\delta^d_t + g) = \alpha A_t \left( \pi \int_0^\zeta \xi^i d\tau + (\varepsilon - \zeta) \psi \right) = \alpha A_t \left( \varepsilon \psi + \int_0^\zeta (\pi \xi^i - \psi) d\tau \right).
\]

Suppose that we let \( q^l_t \) denote the present value of the labor claim (i.e., a claim that pays \( l_{t,s} w_{t,s} \)) and \( q^d_t \) the value of a claim that pays \( \alpha A_s e^{-\int_t^s \delta_t^d \delta_t^d dt} \). Then we obtain the following result.

**Lemma 4** The measure \( \zeta \) of agents choosing the risky options is decreasing in \( \frac{q^d_t}{q^l_t} \). Accordingly, \( \eta^d_t \equiv \delta^d_t + g = \varepsilon \psi + \int_0^\zeta (\pi \xi^i - \psi) d\tau \) is declining in \( \frac{q^d_t}{q^l_t} \).
Lemma 4 is intuitive. Since the entrepreneur is risk averse, she is more concerned with the possibility that the risky project may fail (the downside) than with the prospect of its success. An increasing ratio \( \frac{q_d}{q_l} \) increases the welfare loss conditional on failure compared to the utility of the upside. Accordingly, the entrepreneur prefers the safer option of starting a less productive, but more secure firm.

From this point onwards the determination of equilibria in this economy can proceed as in previous sections with only minor modifications. As before, \( \hat{C}_t = \hat{Y}_t = 1 \) and equation (7) becomes

\[
\beta + g - r_t = \beta \left[ \eta^l q_t^l + \eta^d q_t^d \right]
\]

with \( \eta^l \equiv \delta^l + g \). We wish to focus on equilibria without bubbles, which implies that \( C_t = \beta \left( \sum_{i=d,l} q_i^l \right) Y_t \), or \( \frac{1}{\beta} = \sum_{i=d,l} q_i^l \), since \( C_t = Y_t \). The following lemma is useful for our purposes.

**Lemma 5** Suppose that \( q_t^d \) solves the differential equation

\[
\dot{q}_t^d + \alpha = q_t^d \left[ r_t - g + \eta^d \left( \frac{q_t^d}{q_l} \right) \right]
\]

Define \( q_t^l = \frac{1}{\beta} - q_t^d \). Then

\[
\dot{q}_t^l + 1 - \alpha = q_t^l \left[ r_t - g + \eta^l \right].
\]

Lemma 5 allows us to focus on a single ordinary differential rather than a system of ODEs. In particular, it states that if \( q_t^d \) solves (22), then \( q_t^l = \frac{1}{\beta} - q_t^d \) solves the corresponding asset-pricing equation (23). Plugging (21) into (22) and re-arranging,

\[
\dot{q}_t^d = A \left( q_t^d \right),
\]
Figure 4: Illustration of Proposition 4.

where

\[
A(q_t^d) \equiv q_t^d (\beta + \eta(q_t^d)) - \beta \eta(q_t^d) (q_t^d)^2 - \alpha \quad \text{(24)}
\]

\[
\eta(q_t^d) \equiv \eta^d \left( \frac{q_t^d}{q_t^l} \right) - \eta' = \eta^d \left( \frac{\beta q_t^d}{1 - \beta q_t^d} \right) - \eta'. \quad \text{(25)}
\]

We have the following proposition.

**Proposition 3** There always exists (sufficiently decreasing) \( \eta(q_t^d) \) such that \( A(q_t) \) has at least three positive roots \( q_1, q_2, \) and \( q_3 \) with the properties: a) \( A'(q_1) > 0, \) \( A'(q_2) < 0, \) \( A'(q_3) > 0, \) b) \( q_1 < q_2 < \frac{1}{\beta} \).

Figure 4 illustrates Proposition 4. The figure shows that that \( \dot{q}_t \) is positive between \( q_1 \) and \( q_2 \) and negative between \( q_2 \) and \( q_3 \).

The figure contains a specific numerical example of \( \eta^d \) that satisfies all the properties of Proposition 4. As in the previous section, the steady state \( q_2 \) is stable for \( q_t \in (q_1, q_2) \) and our previous discussion of indeterminacy applies. However, an important difference is that in this example \( q^d \) and \( q^l \) reflect exclusively fundamental values. Moreover, in this numerical example on can verify that the (stable) steady state \( q_2 \) is associated with \( r > g \).
This allows us to conclude that indeterminacy of the components of wealth does not require the presence of bubbles.

From this point on, one can proceed to construct stochastic equilibria of the model driven by extrinsic uncertainty. We do that in the next section after introducing recursive preferences, so that we can discuss the implications of the model for risk premia at the same time.

5 Stochastic equilibria, Recursive preferences and Risk premia

In this section we construct stochastic equilibria driven by extrinsic uncertainty similar to Section 3.2. We also show that when investors have recursive preferences (rather than expected utility preferences), this extrinsic uncertainty carries a risk premium.

To start, we modify one aspect of our earlier analysis by assuming that investors have unit intertemporal elasticity of substitution, but a risk aversion that differs from one. Specifically, we assume the investors’ instantaneous utility flow is no longer given by \( \log (c_{t,s}) \) but rather by the aggregator

\[
f (c_{t,s},V_{t,s}) = (1 + \alpha V_{t,s}) \left( \log c_{t,s} - \frac{\beta}{\alpha} \log (1 + \alpha V_{t,s}) \right),
\]

where \( V_t \) is a consumer’s value function and \( \alpha \) controls the risk aversion of the investor. Utilities of this form are discussed extensively in Schroeder and Skiadas (1999). They correspond to the continuous-time limit of Epstein-Zin-Weil utilities with unit elasticity of substitution. As is highlighted in Schroeder and Skiadas (1999), a convenient transformation is given \( \tilde{V}_{t,s} \equiv \frac{1}{\alpha} \log (1 + \alpha V_{t,s}) \), resulting in

\[
\tilde{V}_{t,s} = E_t \int_t^\infty e^{-\beta(u-t)} \left( \log (c_{u,s}) du + \frac{\alpha}{2} d \left[ \tilde{V} \right]_{u,s} \right),
\]

where \( d \left[ \tilde{V} \right]_{u,s} \) is the time-\( u \) quadratic variation of an investor’s value function, who was born at time \( s \). As is shown in Schroeder and Skiadas (1999), with these preferences the
log-SDF \( \log(m_t) \) follows the dynamics

\[
d \log(m_t) = \alpha \left( \log c_{t,s} - \beta \tilde{V}_{t,s} \right) dt - \beta + \alpha d\tilde{V}_{t,s} - d \log c_{t,s}.
\] (28)

The key feature of (28) is that risk premia are associated not only with variations in the current consumption flow; variations in anticipated long run consumption growth enter the stochastic discount factor as well.

As in Section 3.2, we assume the existence of a diffusion \( x_t \in (0, 1) \) that acts as a coordination device. Before obtaining the main result, we introduce a few definitions

**Definition 2** For two constants \( \eta^{(1)} > 0 \) and \( \eta^{(2)} < 0 \) define

\[
p^{(j)} \equiv \frac{(\beta + \eta^{(j)}) - \sqrt{(\beta + \eta^{(j)})^2 - 4\beta \eta^{(j)} \alpha}}{2 \beta \eta^{(j)}}, \quad j \in \{1, 2\}
\]

It is straightforward to show that \( p^{(1)} < p^{(2)} \).

**Proposition 4** For sufficiently small \( \varepsilon_1, \varepsilon_2 > 0 \), take \( \overline{q} = p^{(1)} + \varepsilon_1, \overline{\overline{q}} = p^{(2)} - \varepsilon_2 \) so that \( \overline{q} < \overline{\overline{q}} \). Then for every downward sloping function \( \tilde{\eta}(q_t) \) with support in \( [\eta^{(2)}, \eta^{(1)}] \), there exists a stationary diffusion \( x_t \) with support in \( [0, 1] \), and a stochastic equilibrium with \( q_t^d = q^d(x_t) \).

In such an equilibrium a) \( q^d(x_t) \) is an increasing function with support in \( [\overline{q}, \overline{\overline{q}}] \), b) \( q^l(x_t) = \frac{1}{\beta} - q^d(x_t) \), c) the interest rate is given by (21), and d) if \( \gamma < 0 \), then the stock market carries a positive equity premium for all \( x_t \).

The next section illustrates proposition 4 by explicitly constructing a stochastic equilibrium and evaluating its quantitative implications.

### 6 Quantitative Evaluation

There are countless possibilities for constructing stochastic equilibria. We illustrate an explicit construction in this section and evaluate it quantitatively.

We start with a few observations. First, we note that in a stochastic equilibrium, the
function $q^d (x_t)$ must satisfy the asset pricing equation

$$
\frac{1}{2} \sigma_x^2 q_{xx}^d + q_x^d \frac{q_x^d}{q^d} \mu_x + \alpha \frac{q_x^d}{q^d} = r_t - g + \eta_t^d + \kappa_t \frac{q_x^d}{q^d} \sigma_x,
$$

(29)

where $\kappa_t = \kappa (x_t)$ is the equilibrium market price of risk. Equation (29) is quite similar to (22), except for the presence of Ito term $\frac{1}{2} \sigma_x^2 q_{xx}^d$ and the equity premium term $\kappa_t \frac{q_x^d}{q^d} \sigma_x$, which arise because $x_t$ is random. Using (21) inside (29) and also $q_t^d = \frac{1}{\beta} - q_t^d$ allows us to express (29) as

$$
\frac{1}{2} \sigma_x^2 q_{xx}^d + q_x^d \frac{q_x^d}{q^d} \mu_x + \alpha \frac{q_x^d}{q^d} = \beta + \eta_t^d - \beta \eta_t^d q_t^d + \kappa_t \frac{q_x^d}{q^d} \sigma_x.
$$

(30)

In the appendix we show that $\kappa_t$ is given by $-\gamma Z(x_t) \sigma_x$ where $Z(x_t)$ solves the differential equation

$$
\frac{Z_{xx}}{2} \sigma_x^2 + \mu_x Z_x - \beta Z - \beta \eta_t q_t^d (x) + \frac{\gamma}{2} Z_x^2 \sigma_x^2 = 0.
$$

(31)

From this point onwards, determining an equilibrium is simply a matter of solving the system of differential equations (30)-(31). This task is greatly simplified if instead of choosing $\mu_x$, we postulate a functional form for $q_t$ and then determine $\mu_x$ so that (30) holds. This is always possible as long as $q_t^d > 0$. A particularly attractive choice is to chose $q (x_t)$ to be log-linear, i.e., $q (x_t) = \bar{q} e^{\zeta x_t}$, where $\zeta = \log (\bar{q}) - \log (\bar{q})$ and then determine $\mu_x$ as

$$
\mu_x = \frac{1}{\zeta} \left[ \beta + \eta_t (x_t) - \beta \eta_t q_t^d (x_t) - \frac{\alpha}{q_t^d (x_t)} - \gamma \zeta Z(x_t) \sigma_x^2 - \frac{1}{2} \sigma_x^2 \zeta^2 \right].
$$

(32)

Since $\bar{q} > p^{(1)}$, $\bar{q} < p^{(2)}$ it is easy to show that $\mu_x (0) > 0$, $\mu_x (1) < 0$. Moreover, as long as $\left| \frac{2 \mu_x (x)}{\sigma_x^2} \right| > 1$ for $x = 0$ and $x = 1$, $x_t$ is stationary. (As we discuss in the appendix, even if this condition is not satisfied, it is always possible to modify $q (x_t)$ to be log-cubic close to the boundaries to ensure that this condition holds).

By chosing $\mu_x$ as in (32), equation (30) automatically holds for $q (x_t) = \bar{q} e^{\zeta x_t}$. Accordingly, rather than solving a system of differential equations, one can simply substitute the expression for $\mu_x$ (equation (32)) inside (31) and solve a single ordinary differential equation
<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average aggregate consumption growth rate</td>
<td>2.3 %</td>
<td>2.3 %</td>
</tr>
<tr>
<td>Volatility of aggregate consumption growth rate</td>
<td>3.3 %</td>
<td>0 %</td>
</tr>
<tr>
<td>Individual agents’ annual volatility of consumption growth</td>
<td></td>
<td>0.71%</td>
</tr>
<tr>
<td>Sharpe ratio</td>
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<td>0.29</td>
</tr>
<tr>
<td>Stock market volatility</td>
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<td>20%</td>
</tr>
<tr>
<td>Equity premium</td>
<td>5.2%</td>
<td>5.82%</td>
</tr>
<tr>
<td>Interest rate</td>
<td>2.8%</td>
<td>0.9%</td>
</tr>
<tr>
<td>Standard deviation of the interest rate</td>
<td>0.92%</td>
<td>0.71%</td>
</tr>
<tr>
<td>(log) Price-Dividend ratio</td>
<td>3.206</td>
<td>3.40</td>
</tr>
<tr>
<td>Standard deviation of the log Price-Dividend ratio</td>
<td>0.27</td>
<td>0.36</td>
</tr>
<tr>
<td>Autocorrelation of the log Price-Dividend ratio</td>
<td>0.89</td>
<td>0.91</td>
</tr>
<tr>
<td>Dividend growth volatility</td>
<td>11.2%</td>
<td>6.56%</td>
</tr>
</tbody>
</table>

Table 1: Unconditional moments for the data and the model. The data for the average equity premium, the volatility of returns, and the level of the interest rate are from the long historical sample available from the website of R. Shiller (http://www.econ.yale.edu/?shiller/data/chapt26.xls). The volatility of the ?real rate? is inferred from the yields of 5-year constant maturity TIPS.

for $Z (x_t)$.  

To provide a simple numerical illustration, we start by fixing a few parameters. Specifically, we choose the effective subjective discount rate to be $\beta = 0.02$ and $\eta^l = 0.04$. The choice $\eta^l = 0.04$ is motivated by simplicity: it implies a wage that neither increases nor declines over the life of a worker. For the fraction of output that accrues to labor we choose $1 - \alpha = 0.6$. The function $\eta (x_t) = \eta^d (x_t) - \eta^l$ is determined as $\eta (x_t) = \eta^{(2)} x^p + \eta^{(1)} (1 - x^p)$, where $\eta^{(2)} = -0.04$, $\eta^{(1)} = 0.3$, $p = 0.6$. With this functional form the equilibrium average growth rate (per capita) of dividends of the market portfolio turns out to be approximately the same as in the data, and the annual volatility of dividend growth is about half of the volatility of dividend growth in the data, allowing us to illustrate that our results are driven by valuation changes rather than cash flow variation. The final choice concerns the range $[\tilde{q}, \bar{q}]$. The determination of this range is somewhat arbitrary and we choose it so that the Price-to-earnings ratio stays in the range $[5, 40]$. This choice is admittedly arbitrary, but as we explain in the appendix, the exact choice of these values is immaterial as long as we

---

3One could even postulate a specific functional form for $Z (x_t)$ and solve for $\bar{\eta}(x_t)$ to ensure that (31) holds. However, it is not simple to determine conditions on allowable functional forms of $Z (x_t)$ so that $\bar{\eta}(x_t)$ is declining.
<table>
<thead>
<tr>
<th>Year</th>
<th>$\beta$ (Data)</th>
<th>$\beta$ (Model)</th>
<th>$R^2$(Data)</th>
<th>$R^2$(Model)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.130</td>
<td>-0.186</td>
<td>0.040</td>
<td>0.088</td>
</tr>
<tr>
<td></td>
<td>[-0.578, 0.049]</td>
<td></td>
<td>[0.000, 0.274]</td>
<td></td>
</tr>
<tr>
<td>3</td>
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<td>-0.405</td>
<td>0.090</td>
<td>0.190</td>
</tr>
<tr>
<td></td>
<td>[-1.017, 0.148]</td>
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<td></td>
</tr>
<tr>
<td>5</td>
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<td>-0.512</td>
<td>0.180</td>
<td>0.239</td>
</tr>
<tr>
<td></td>
<td>[-1.124, 0.272]</td>
<td></td>
<td>[0.001, 0.548]</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>-0.750</td>
<td>-0.568</td>
<td>0.230</td>
<td>0.261</td>
</tr>
<tr>
<td></td>
<td>[-1.133, 0.429]</td>
<td></td>
<td>[0.001, 0.560]</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Long-horizon regressions of excess returns on the log P/D ratio. The simulated data are based on 1000 independent simulations of 100-year long samples. For each of these 100-year long simulated samples, we run predictive regressions of the form \( \log R^e_{t→t+h} = \alpha + \beta \log(P_t/D_t) \), where \( \log R^e_{t→t+h} \) denotes the time-t gross excess return over the next h years. We report the mean values for the coefficient $\beta$ and the $R^2$ of these regressions, along with the respective [0.025, 0.975] percentiles.

adapt the other parameters (especially the volatility) to keep the stationary distribution of the price-to-earnings ratio unaffected. We choose $\sigma$ to match the volatility of the stock market. Finally, we choose the risk aversion parameter $|\gamma| = 5$, which is well within the range of reasonable risk aversion values. Table 1 provides a comparison between the model-implied unconditional moments and the respective moments in the data. In reporting the results we follow the approach of Barro (2006) to relate the results of our model (which produces implications for an all-equity financed firm) to the data (where equity is levered). Specifically, we use the well known Modigliani-Miller formula relating the returns of levered equity to those of unleveled equity, along with the historically observed debt-to-equity ratio, to report model-implied levered returns. (Specifically, we set the ratio of levered to unleveled equity returns to be equal to 1.65, as in Barro(2006)).

Inspection of table 1 shows that the model is quite successful at reproducing all the asset pricing moments. To put these numbers in the proper relation to the literature, it is worth highlighting that a) agents in this model have unitary IES and a very moderate risk aversion c) the interest rate has very low volatility, d) aggregate consumption growth is constant and d) the volatility of dividend growth is about half the respective value in the data.

To underscore the model’s success in driving fluctuations in asset prices from valuation

\[A \text{ more aggressive choice is to choose that parameter as the ratio of consumption growth volatility to dividend growth volatility, which results in values around 3. See, for instance Bansal and Yaron (2004).}\]
Figure 5: Calibration results. Equity premium, market price of risk (Sharpe ratio), interest rate, and stock return volatility for the baseline parametrization as a function of the state variable ($x_t$). The range of values of $x_t$ correspond to the interval between the interquartile range of $x_t$ (0.25 to 0.75 percentile) of the stationary distribution of $x_t$.

effects, table 2 reports results of simulated predictability regressions inside the model and compares the results with the data. The main takeaway of the table is that the model-implied predictability is strikingly close to the respective time-variation in the data. The reason for the quantitative success of the model is that it considers a very different set of shocks than the ones that are typically studied in the literature. The typical approach in the literature is to model the aggregate endowment process as being exogenous and then make asset prices endogenous. The distinguishing feature of the present model is that both prices and individual consumption processes are jointly endogenous to fluctuations in expectation formation. Hence, one can start by determining expectations fluctuations that imply realistic asset price dynamics and then check that these choices don’t imply unrealistic fluctuations in dividend growth or consumption growth of individual investors.

In that vein we note that the annual volatility of an individual’s annual (time-integrated)
consumption growth in the present model is only 70 basis points. Hence, both dividend and consumption growth rates exhibit – if anything– a substantially lower annual, time-integrated volatility than in the data. In short, the model shows that the volatile behavior of asset prices can be matched without making the “real side of the model” too volatile.

A noteworthy property of the present model is that its quantitative success does not rely on substantial annual volatility of the cross sectional variance of log consumption, nor does it require that the log price-dividend ratio and the cross sectional variance of log consumption have similar persistence. Figure illustrates the point. The figure plots a sample path of the cross sectional variance of log consumption over a sample that is similar in length to the post-war sample. It also plots the log price-dividend ratio (on the right scale). There are a few patterns that emerge from this graph, which are consistent with the data. The year-over-year changes in inequality (as measured by the cross sectional variance of log consumption) are very small compared to the time-variation of the log-price dividend ratio. Inequality is positively correlated with the price-dividend ratio at high frequencies, but the two time-series have very different low-frequency variation, with inequality being substantially more

Figure 6: A simulated path of the cross-sectional variance of log consumption and the log price-dividend ratio.
These patterns are very much consistent with the data. In the data, consumption inequality only changes by a few basis points on a yearly basis. Moreover, in the data inequality measures behave like unit root processes, consistent with figure. These features distinguish this model from Constantinides and Duffie (1996), where the mechanism relies on the variation of cross-sectional inequality and also from models that rely on (preference-, belief-, or investment opportunity-) heterogeneity, which tend to imply a tight link of the price dividend ratio with consumption inequality.

The main conclusion we wish to draw is that the “excess volatility” of asset prices does not translate into excess volatility of the real side of the model. If anything the model under predicts annual dividend and consumption volatility, and implies movements in the cross-sectional variance of log consumption that are even smoother than in reality.

\[ \text{For instance, Krueger and Perri report that the cross sectional variance of log consumption changed from about 0.18 to about 0.23 from 1980 to 2003.} \]